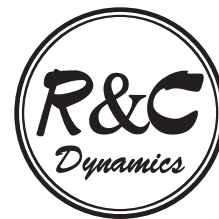**A. V. BORISOV, I. S. MAMAEV, A. A. KILIN**

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ABSOLUTE AND RELATIVE CHOREOGRAPHIES IN THE PROBLEM OF POINT VORTICES MOVING ON A PLANE

We obtained new periodic solutions in the problems of three and four point vortices moving on a plane. In the case of three vortices, the system is reduced to a Hamiltonian system with one degree of freedom, and it is integrable. In the case of four vortices, the order is reduced to two degrees of freedom, and the system is not integrable. We present relative and absolute choreographies of three and four vortices of the same intensity which are periodic motions of vortices in some rotating and fixed frame of reference, where all the vortices move along the same closed curve. Similar choreographies have been recently obtained by C. Moore, A. Chenciner, and C. Simo for the n-body problem in celestial mechanics [6, 7, 17]. Nevertheless, the choreographies that appear in vortex dynamics have a number of distinct features.

Equations of motion and first integrals. Let us review briefly basic forms of equations and first integrals in the dynamics of point vortices on a plane (for a detailed discussion, see [4, 14, 16], where, in addition, hydrodynamical assumptions required for the validity of these equations are specified).

For n point vortices with Cartesian coordinates (x_i, y_i) and intensities Γ_i, the equations of motion can be written in Hamiltonian form,

$$\Gamma_i \dot{x}_i = \frac{\partial \mathscr{H}}{\partial y_i}, \quad \Gamma_i \dot{y}_i = -\frac{\partial \mathscr{H}}{\partial x_i}, \quad 1 \leqslant i \leqslant n, \tag{1}$$

where the Hamiltonian is

$$\mathscr{H} = -\frac{1}{4\pi} \sum_{i<j}^{n} \Gamma_i \Gamma_j \ln |\boldsymbol{r}_i - \boldsymbol{r}_j|^2, \quad \boldsymbol{r}_i = (x_i, y_i). \tag{2}$$

Here, the Poisson bracket is:

$$\{f, g\} = \sum_{i=1}^{N} \frac{1}{\Gamma_i} \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial x_i} \right). \tag{3}$$

Equations (1) are invariant under the action of the Eucledian group $E(2)$, so (in addition to the Hamiltonian) they also have three first integrals

$$Q = \sum_{i=1}^{n} \Gamma_i x_i, \quad P = \sum_{i=1}^{n} \Gamma_i y_i, \quad I = \sum_{i=1}^{n} \Gamma_i (x_i^2 + y_i^2), \tag{4}$$

which, however, are not involutive,

$$\{Q, P\} = \sum_{i=1}^{N} \Gamma_i, \quad \{P, I\} = -2Q, \quad \{Q, I\} = 2P. \tag{5}$$

Using these three integrals, it is possible to construct two involutive integrals, $Q^2 + P^2$ and I. Hence, according to the general theory [10, 13], we can reduce the number of degrees of freedom by two. Thus, in the particular case of three vortices, the system can be reduced to one degree of freedom and becomes integrable (Gröbli, Kirchhoff, Poincaré) [4, 14, 16, 18], while the problem of four vortices is reduced to a system with two degrees of freedom. Generally, the last problem is not integrable [20].

Effective reduction in the system of four vortices with intensities of the same sign was done by K. M. Khanin in [9]. In that work two pairs of vortices were considered and for each pair action-angle variables were selected. The system of four vortices was then obtained as a perturbation of the two two-vortex systems. He proves (using the methods of KAM-theory) the existence of quasiperiodic solutions. As a small parameter, he takes a value inverse to the distance between two pairs of vortices.

Reduction by one degree of freedom using the translational invariants P and Q was done by Lim in [11]. He introduced barycentric Jacobi coordinates (centered, in this case, at the center of vorticity), which have well-known analogs in the classical n-body problem in celestial mechanics [5]. Note that even this (partial) reduction made it possible to apply some methods of KAM-theory to the problem of point vortices' motion [11, 12].

When n vortices on a plane have equal intensities, one of the more formal methods of order reduction, which was described in [2] (see also [3, 4]), proved to be most suitable. This method is based on using mutual variables representation of the equations of motion. As mutual variables, the squares of distances between the pairs of vortices and oriented areas of triangles were taken:

$$M_{ij} = (x_i - x_j)^2 + (y_i - y_j)^2, \quad \Delta_{ijk} = (\boldsymbol{r}_j - \boldsymbol{r}_k) \wedge (\boldsymbol{r}_k - \boldsymbol{r}_i). \tag{6}$$

Mutual commutation of such variables (which are introduced by E. Laura) leads to some Lie algebra. Here, the reduction procedure (more exactly, the last canonical step of it) is equivalent to a purely algebraic problem of introduction of symplectic coordinates on the orbits of corresponding Lie algebras. (For identical vortices the method of reduction based on the Fourier transformatio was introduced also in [1].)

Reduction for three and four vortices of equal intensity. Without losing in generality, we put $\Gamma_i = \Gamma_j = 1$, $P = Q = 0$, then the moment integral I (4) can be written as

$$I = \frac{1}{n} \sum_{i<j}^{n} M_{ij}, \tag{7}$$

where n is the number of vortices. Using the complex representation of the vortices' coordinates, $z_k = x_k + iy_k$, we obtain:

$$z_k = \frac{1}{n} \sum_{j=k}^{n} \sqrt{M_{kj}} e^{i\theta_{kj}}, \tag{8}$$

where θ_{kj} is the angle between the vector from the j-th vortex to the k-th vortex and the positive direction of Ox-axis.

The following propositions, defining the dynamics of a reduced system of vortices, can be obtained by direct computation:

Proposition 1. *For three vortices of equal intensity, evolution of the mutual distances* (6) (*assuming $I = $ const*) *is described by a Hamiltonian system with one degree of freedom. In canonical*

variables (g, G), the system looks as

$$\dot{g} = \frac{\partial \mathscr{H}}{\partial G}, \quad \dot{G} = -\frac{\partial \mathscr{H}}{\partial g}, \quad \mathscr{H} = -\frac{1}{4\pi} \ln M_{12} M_{13} M_{23}, \tag{9}$$

where $M_{12} = 4\left(\frac{I}{2} - G\right)$, $M_{13} = 8G - I + 2\sqrt{12}\sqrt{\left(\frac{I}{2} - G\right)G}\cos g$, $M_{23} = 4\left(\frac{I}{2} - G\right) -$
$- 2\sqrt{12}\sqrt{\left(\frac{I}{2} - G\right)G}\cos g.$

Proposition 2. *For four vortices of equal intensity, evolution of the mutual distances is described by a Hamiltonian system with two degrees of freedom. In canonical variables (g, G, h, H), the system looks as*

$$\dot{g} = \frac{\partial \mathscr{H}}{\partial G}, \quad \dot{G} = -\frac{\partial \mathscr{H}}{\partial g}, \quad \dot{h} = \frac{\partial \mathscr{H}}{\partial H}, \quad \dot{H} = -\frac{\partial \mathscr{H}}{\partial h}, \tag{10}$$

$$\mathscr{H} = -\frac{1}{4\pi} \ln M_{12} M_{13} M_{14} M_{23} M_{24} M_{34},$$

where

$$M_{12} = I - G + 2\sqrt{(I - H)(I - G)}\cos h, \qquad M_{34} = I - G - 2\sqrt{(I - H)(I - G)}\cos h,$$
$$M_{13} = I + G + 2\sqrt{(I - H)G}\cos(h + g), \qquad M_{24} = I + G - H - 2\sqrt{(I - H)G}\cos(h + g),$$
$$M_{14} = H + 2\sqrt{(H - G)G}\cos g, \qquad M_{23} = H - 2\sqrt{(I - G)G}\cos g.$$

REMARK. These canonical variables are of natural geometrical origin related to representation of the equations of motion on a Lie algebra [2, 3].

Absolute motion: quadratures and geometric interpretation. According to (8), when $M_{ij}(t)$ are known, one needs to know the angles $\theta_{ij}(t)$ to determine the vortices' coordinates. It is obvious that only one of the angles is independent (in this case, we take θ_{12} to be independent), the remaining angles are computed with the cosine theorem:

$$\theta_{ij} + \theta_{ik} = \arccos\left(\frac{M_{jk} - M_{ij} - M_{ik}}{2\sqrt{2M_{ij}M_{ik}}}\right), \quad i \neq j, \, k \neq i. \tag{11}$$

Evolution of θ_{12} is obtained by quadrature [14]:

$$4\pi\dot{\theta}_{12} = M_{13}^{-1} + M_{23}^{-1} + M_{12}^{-1}(6 - M_{23}^{-1}M_{13} - M_{13}M_{23}^{-1}) \quad \text{(for three vortices)},$$
$$4\pi\dot{\theta}_{12} = M_{13}^{-1} + M_{14}^{-1} + M_{23}^{-1} + M_{24}^{-1} + \tag{12}$$
$$+ M_{12}^{-1}(8 - M_{13}^{-1}M_{23} - M_{14}^{-1}M_{24} - M_{23}^{-1}M_{13} - M_{24}^{-1}M_{14}) \quad \text{(for four vortices)}.$$

In the case of periodic solutions of the reduced system (9), (10), there is an interesting geometric interpretation of the absolute motion.

Proposition 3. *Let $\gamma(t)$ be a periodic solution (of period T) of the reduced system, then*
$1°$ *there exists a frame of reference, uniformly rotating with some angular velocity Ω_a about the center of vorticity in which each vortex moves along some closed curve $\xi_i(t)$;*
$2°$ *the rotational velocity Ω_a is given by (to within $\frac{2\pi}{T}\frac{p}{q}$, $p, q \in \mathbb{Z}$):*

$$\Omega_a = \frac{1}{T} \int_0^T \dot{\theta}_{12}(t)\,dt. \tag{13}$$
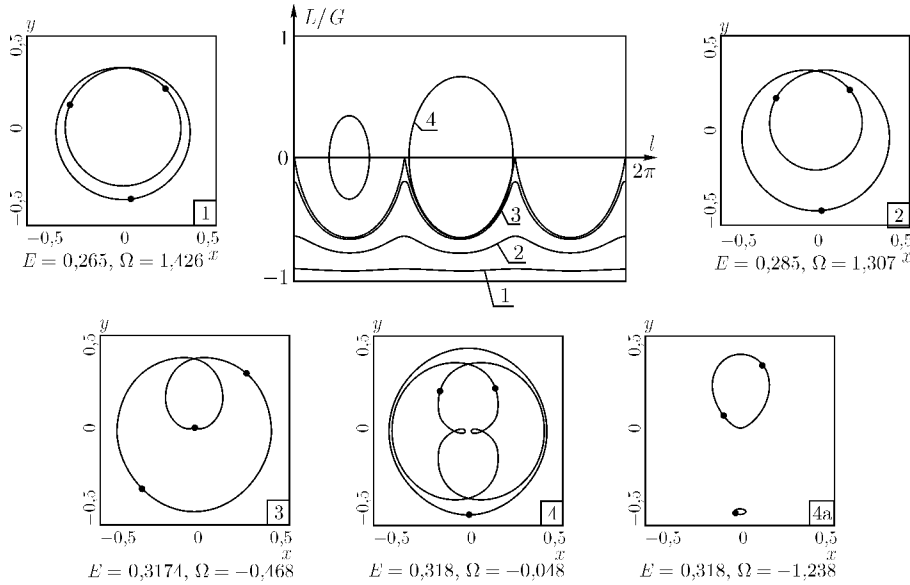
Fig. 1. The phase portrait of the reduced system in the case of three vortices. Relative choreographies corresponding to different phase trajectories on the portrait

3° *if the velocities Ω_a and $\Omega_o = \dfrac{2\pi}{T}$ are commensurable (i. e. $\dfrac{\Omega_a}{\Omega_o} = \dfrac{p}{q}$, $p, q \in \mathbb{Z}$), then the vortices in the fixed frame also move along some closed curves;*

4° *if any of the curves $\xi_i(t)$ can be superimposed by rotation about the center of vorticity by an angle, commensurable with 2π, then there exists a (rotating) frame of reference such that the corresponding vortices move along the same curve.*

Proof.

We expand the periodic function of period T in the right-hand side of (12) in a convergent Fourier series

$$4\pi\dot{\theta}_{12} = \sum_{n\in\mathbb{Z}} a^{(n)} e^{i\frac{2\pi n}{T}t}. \tag{14}$$

Integrating (14) while bearing (11) in mind, we come to a conclusion that the angles θ_{ij} depend on t in the following way:

$$\theta_{ij}(t) = \Omega_{ij}t + g_{ij}(t), \tag{15}$$

where $\Omega_{ij} = a^{(0)} + \dfrac{2\pi}{T}\dfrac{q_{ij}}{p_{ij}}$, $q_{ij}, p_{ij} \in \mathbb{Z}$, while $g_{ij}(t) = g_{ij}(t + T)$ are T-periodic functions of time.

Substituting this into (8), we see that locations of the vortices on a plane is given as follows:

$$z_k(t) = \frac{Q + iP}{\sum \Gamma_i} + u_k(t)e^{i\Omega t}, \quad u_k(t) = u_k(t + T_1) \in \mathbb{C}, \quad \Omega = a^{(0)}. \tag{16}$$

Hence, *in the frame of reference, rotating about the center of vorticity with angular velocity Ω, all the vortices move along closed analytical curves, given by functions $u_k(t) \in \mathbb{C}$.*

The proof of 2°, 3°, 4° with (14)–(16) is evident. ∎

REMARK. Proposition 3 is generalized without change to the case of arbitrarily many vortices, n, provided that $\gamma(t)$ is a periodic solution of the reduced system with $2n - 2$ degrees of freedom. (For a more detailed discussion of reduction, see, for example, [4].)

Analytical choreographies. Now we show that the four- and three-vortex problems can have remarkable periodic solutions, when all the vortices follow each other along the same curve; such solutions are referred to as simple (or connected) choreographies. To underline the difference between choreographies in a fixed frame of reference and choreographies in a rotating one, these choreographies are called *absolute* and *relative* [7].

Theorem 1 ([4]). *If in the problem of three vortices of equal intensity, the constants of the integrals of motion, I and $\mathscr{H}$, satisfy the inequality*

$$-\ln 3 < \frac{4\pi}{3}\mathscr{H} + \ln I < \ln 2, \tag{17}$$

then this motion is a simple relative choreography (see Fig. 1).

Proof.

Since in the case of three vortices, the reduced system (9) has one degree of freedom, all $M_{ij}(t)$ are periodic functions of the same period T. It can be easily shown that under the restriction (17), the orientation of the vortex triangle is unchanged, and there are times t_1 and t_2 such that $M_{23}(t_1) = M_{13}(t_2)$. (If (17) is not met, then $\forall t\ \exists k$ such that $M_{ij}(k) < M_{ik}(t)$ and $M_{ij}(k) < M_{jk}(t)$.) Moreover, since with fixed H, I all M_{ij} are expressed through one of them (for example, M_{12}), the following relations also hold true:

$$M_{13}(t_1) = M_{12}(t_2), \quad M_{12}(t_1) = M_{23}(t_2).$$

Since the equations are invariant under cyclic permutation of the vortices (as implied by equality of the intensities), we find that

$$t_1 - t_2 = \frac{T}{3}n, \quad n \in \mathbb{Z}.$$

Since the evolution equation for M_{ij} is of the first order, we conclude that

$$M_{12}(t) = M_{23}\left(t + \frac{T}{3}\right) = M_{13}\left(t + \frac{2T}{3}\right) = f(t),$$

or

$$M_{12}(t) = M_{23}\left(t + \frac{2T}{3}\right) = M_{13}\left(t + \frac{T}{3}\right) = f(t),$$

where $f(t)$ is some T-periodic function.

Substituting this into (8), we obtain

$$z_k(t) = u\left(t + \frac{k-1}{3}T\right)e^{i\Omega t}, \quad k = 1, 2, 3,$$

where $u(t)$ is a T-periodic complex-valued function, determining the same curve, along which the vortices move, in the frame of reference rotating with angular velocity Ω. The phase portrait of the reduced system of the 3-vortex problem and the corresponding relative choreographies are given in Fig. 1. ∎

The 4-vortex problem has an exceptional solution, given by quadratures — Goryachev's solution, where the vortices form a parallelogram at each instant of time [8]. As in the case of the 3-vortex problem, it is quite easy to show that

Theorem 2 ([4]). *If in the 4-vortex problem, the vortices (of equal intensity) form a centrally symmetric configuration (a parallelogram), while the constants $\mathscr{H}$ and I satisfy the inequality*

$$-\ln 2 < \frac{2\pi}{3}\mathscr{H} + \ln I < -\ln\frac{144}{5},$$

then the motion is a simple relative choreography.

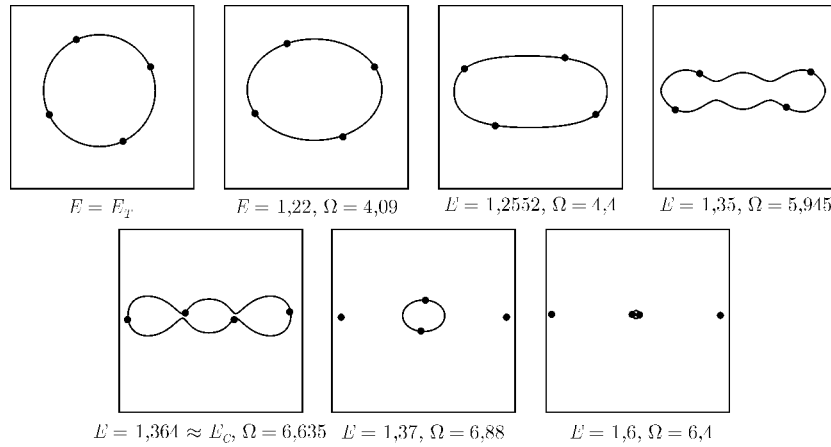The corresponding relative choreographies are shown in Fig. 2.

Fig. 2. Relative choreographies in the 4-vortex problem.

REMARK. The physical meaning of the inequality is as follows: when I is fixed, the type of the motion in the 3-vortex problem and in the case of Goryachev's solution is changed at energy values corresponding to the Thomson and collinear configurations.

New periodic solution in the 4-vortex problem. Now we show that, aside from the choreography just described, the 4-vortex problem has at least one more choreography (different from Goryachev's solution). Consider the vicinity of Thomson's solution, i. e. the motion, where the vortices are located at the vertices of a square and rotate uniformly about the vorticity center [19]. It is obvious that in the case of the reduced system with two degrees of freedom (10), Thomson's solution is represented by a fixed point (more precisely, by six points corresponding to various arrangements of vortices at square's vertices). We consider one of the arrangenents (the rest are completely identical) with coordinates

$$G = 0, \quad H = \frac{1}{2}, \quad h = \frac{\pi}{2}.$$

Let us find the normal form of the reduced system's (10) Hamiltonian near this point. To do that,
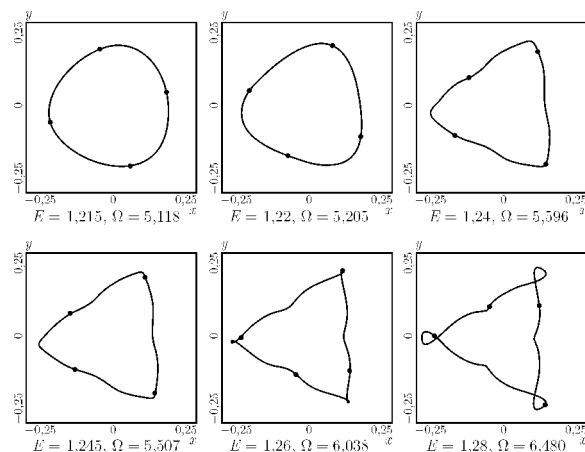

Fig. 3. Relative choreographies corresponding to the new periodic solution in the three-body problem.

we start with a canonical change of variables:

$$G = \frac{x^2 + X^2}{2}, \quad g = \operatorname{arctg} \frac{x}{X}, \quad H = \frac{1}{2} + 8^{-1/4}Y, \quad h = \frac{\pi}{2} + 8^{1/4}y;$$

Expanding the Hamiltonian in a series up to and including quadratic terms, we find

$$\mathscr{H} = \frac{\ln 2}{\pi} + \mathscr{H}_2 + \mathscr{H}_r, \tag{18}$$

$$\mathscr{H}_2 = \frac{1}{4\pi} \left(3(x^2 + X^2) + 2\sqrt{2}(y^2 + Y^2) \right),$$

where the expansion of $\mathscr{H}_r$ starts with the third-order terms. Thus, the Hamiltonian $\mathscr{H}_2$ defines an integrable system with two incommensurable frequencies and has precisely two non-degenerate periodic solutions on each energy level $\mathscr{H}_2 = h_2 = \mathrm{const}$. These solutions are given by

$$x = X = 0, \quad y^2 + Y^2 = \frac{\pi}{\sqrt{2}}h_2, \tag{19}$$

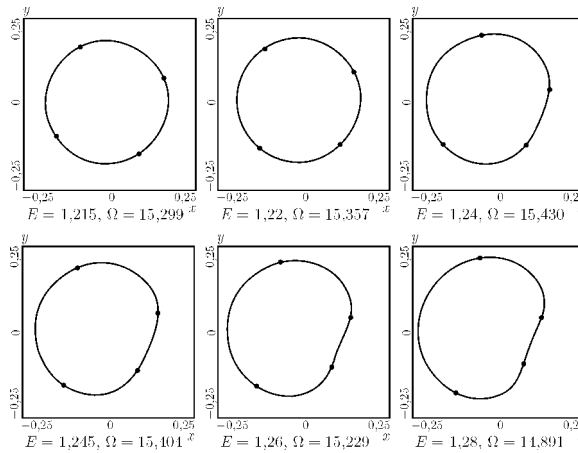$$y = Y = 0, \quad x^2 + X^2 = \frac{4\pi}{3}h_2. \tag{20}$$

Fig. 4. Relative choreographies corresponding to the new periodic solution in the three-body problem in the frame of reference different from that used in Fig. 3.

According to the Lyapunov theorem [15], these solutions are preserved under perturbations, hence, the complete system (18) in the vicinity of the fixed point also has two non-degenerate periodic solutions on each energy level. It is easy to show that one the solutions (corresponding to (19)) is identical to Goryachev's solution — during their motion the vortices are located at the vertices of a parallelogram. At the same time, the other solution (20) does not have such a simple geometric interpretation.

Since equations (1) are invariant under a cyclic permutation of the vortices $\sigma_c(z_1, z_2, z_3) = (z_3, z_1, z_2)$ and the eigenvalues of the Hamiltonian $\mathscr{H}_2$ are different, it is easy to show that both periodic solutions are also invariant under σ_c. Thus, according to Proposition 3, in the appropriate frame of reference, all the vortices move along the same curve, i.e. both solutions correspond to simple relative choreographies. Figure 3 shows the relative choreographies that correspond to the new periodic solution of the reduced system (10).

Relative and absolute choreographies. Generally, for each periodic solution (of period T) of the reduced system (9), (10), it is possible to specify a countable set of rotating frames of reference, where the vortices move along closed curves. Indeed, the trajectories will remain closed in a frame rotating with velocity

$$\Omega'_a = \Omega_a + \frac{p}{q}\frac{2\pi}{T}, \quad p, q \in \mathbb{Z} \tag{21}$$

Nevertheless, the change (21) with arbitrary p and q does not preserve the connectedness of the trajectories, i. e. in the general case, after moving to the frame of reference, rotating with velocity Ω'_a, a simple relative choreography decomposes into separate closed curves, along which the vortices move. To preserve the connectedness, the following criterion should be met.

Proposition 4. *Let a periodic solution (of period T) of the reduced system describe a connected relative choreography, the rotational velocity of the frame of reference being Ω_a, while the period of the motion of the vortices along the corresponding common curve being equal to mT. Then, if*

$$mp = knq, \tag{22}$$

where n is the number of vortices, and $k \in \mathbb{Z}$ is an arbitrary integer, the transformation (21) results in a connected choreography.

Proof.
For the solution in question, the absolute coordinates of the vortices can be presented in the form:

$$z_k(t) = u\left(t + \frac{k-1}{n}mT\right)e^{i\Omega_a t}, \tag{23}$$

where $u(t) = u(t + mT)$ is a periodic complex-valued function (of period mT).

Solving for Ω_a from (21) and substituting into (23), we get

$$z_k(t) = u\left(t + \frac{k-1}{n}mT\right)e^{-i\frac{2\pi}{T}\frac{p}{q}t}e^{i\Omega'_a t} = u_k(t)e^{i\Omega'_a t}. \tag{24}$$

If all the vortices move along the same curve, then their coordinates in the rotating (with velocity Ω'_a) frame of reference are equal to $u_k(t)$ and satisfy $u_{k+1}(t) = u_k\left(t + \frac{mT}{n}\right)$, whence, taking (24) into account, we obtain (22). ∎

The relation (22) is a sufficient but not necessary condition for the choreography to be connected. If the curve along which the vortices move has additional symmetries, then, besides the p, q that meet the condition (22), there are more velocities of the form (21), which result in connected choreographies (see below, for Goryachev's solution).

It is interesting to note that using the transformation (21), some choreographies can be "disentangled" — for example, Fig. 4 shows choreographies, corresponding to the periodic solution (20), in the frame of reference, different from that used in Fig. 3, their rotational velocities differ by

$$\Omega' - \Omega = \frac{4}{3}\frac{2\pi}{T}.$$

As it was shown above (see Proposition 3), if, for a relative choreography, the period T of the reduced system's solution is commensurable with the rotational period $T_a = \frac{2\pi}{\Omega_a}$ of the frame of reference, then in a fixed frame of reference, all the vortices move along closed (and, usually, different) curves.

Let us consider in greater detail the existence of absolute choreographies in the three- and four-vortex problems. According to what was said above, any relative choreography, corresponding to the
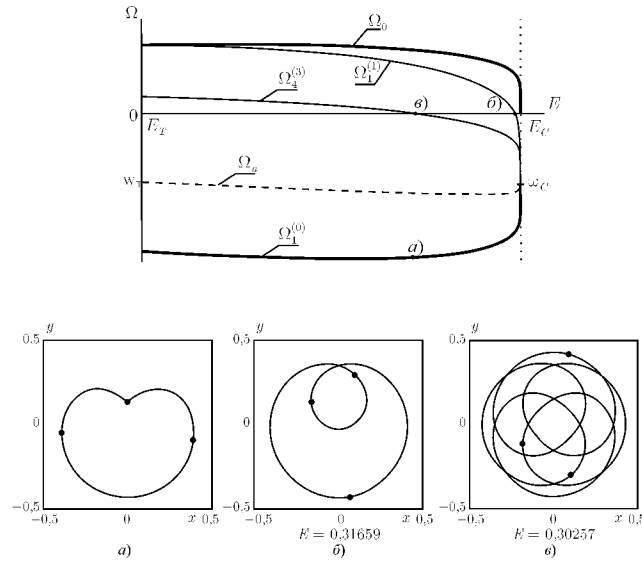
Fig. 5. The rotational angular velocities, $\Omega_1^{(1)}$ and $\Omega_4^{(3)}$, of relative choreographies are presented as functions of energy. The points where the graphs intersect the Ox axis correspond to the absolute choreographies of the three vortices shown in the bottom (Figs. b and c). The heavy lines denote the basic angular velocity $\Omega_1^{(0)}$, corresponding to the simplest relative choreography (shown in Fig. a), and the frequency of the periodic solution of the reduced system (9), $\Omega_0 = \dfrac{2\pi}{T}$.
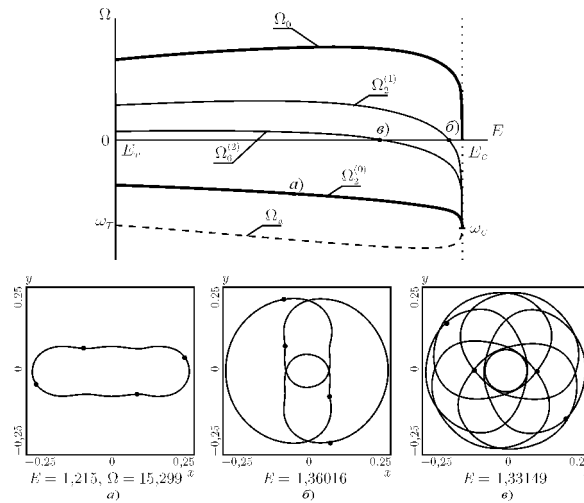

Fig. 6. The rotational angular velocities, $\Omega_2^{(1)}$ and $\Omega_6^{(2)}$, of relative choreographies are presented as functions of energy. The points where the graphs intersect the Ox axis correspond to the absolute choreographies of the four vortices shown in the bottom (Figs. b and c). The heavy lines denote the basic angular velocity $\Omega_2^{(0)}$, corresponding to the simplest relative choreography (shown in Fig. a), and the frequency of the reduced system (10), $\Omega_0 = \dfrac{2\pi}{T}$.

periodic solution (of period T) of the reduced system (see (9), (10)), closes in time mT, $m \in \mathbb{N}$. During this time interval, the vortices pass one and the same relative configuration m times. Let $\Omega_m^{(k)}$ be the rotational velocities of the frames of reference related to these choreographies.

As it was shown above, all the solutions of the reduced three-vortex system, for a fixed D and $E_T < E < E_C$ (where E_T and E_C are the energies corresponding to the Thomson and collinear configurations), describe connected relative choreographies. Moreover, there is a frame of reference, where the choreography closes in (the shortest possible) time T (see Fig. 5); the corresponding angular velocity is denoted as $\Omega_1^{(0)}$, its graph is shown in Fig. 5. The angular velocities of the other relative connected choreographies are now given by

$$\Omega_m^{(k)}(E) = \Omega_1^{(0)}(E) + \frac{3k}{m}\Omega_0(E), \text{ where } m \in \mathbb{N}, \ k \in \mathbb{Z}; \tag{25}$$

here $3k$ and m are coprime numbers, and $\Omega_0(E) = \dfrac{2\pi}{T(E)}$. The velocity $\Omega_m^{(k)}$ corresponds to the choreography that closes in time mT.

Obviously, the absolute choreographies are defined by the solutions of the equation

$$\Omega_m^{(k)}(E) = 0, \tag{26}$$

where k, m are fixed, while E is unknown. Figure 5 shows the graphs of velocities $\Omega_m^{(k)}(E)$ together with the solutions of equation (26), as well as the corresponding absolute choreographies (the simplest choreographies in a three-vortex system). Generally, there is a countable set of absolute choreographies with different m, k.

Indeed, consider the function

$$f_a(E) = \Omega_1^{(0)}(E) + a\Omega_0(E), \quad a \in \mathbb{R},$$

for which the equality $f_a(E_C) = \omega_C < 0$ holds (see Fig. 5). Since $\Omega_0(E) > 0$, there exists a number a_* such that when $a > a_*$, the function $f_a(E)$ has at least one zero in the interval $[E_T, E_C]$. It is clear that the interval $[a_*, +\infty)$ contains an infinite number of rationals of the form $a = \dfrac{3k}{m}$, where $3k$ and m are coprimes. The dotted line in Fig. 5 shows the curve $f_{a^*}(E)$ and the relative choreography's rotational angular velocity Ω_a, calculated from (13). This choreography is the simplest disconnected choreography and is remarkable for the fact that the values at the ends of the interval, $\Omega_a(E_T)$ and $\Omega_a(E_C)$, are equal to the rotational angular velocities of the Thomson and collinear configurations.

For Goryachev's solution in the four-vortex system, the reasoning is similar but slightly modified. First of all, one can show that the simplest connected choreography closes in time $2T$, while the vortices in this case pass one and the same relative configuration twice (i. e. velocities $\Omega_1^{(k)}$ correspond to disconnected choreographies). The graph of one of the corresponding angular velocities, which we denote as $\Omega_2^{(0)}$, is given in Fig. **??**. In this case, due to the symmetry of the curve related to the choreography $\Omega_2^{(0)}$, the frequencies corresponding to connected choreographies must satisfy the relation different from (25),

$$\Omega_{2m}^{(k)}(E) = \Omega_2^{(0)}(E) + \frac{k}{m}\Omega_0(E), \text{ where } m \text{ is odd}, \ k \in \mathbb{Z}; \tag{27}$$

here k and m are coprimes, and $\Omega_0(E) = \dfrac{2\pi}{T(E)}$, where T is the period of Goryachev's solution to the reduced system (10). This choreography closes in time $2mT$.

As above, the equation

$$\Omega_{2m}^{(k)}(E) = 0$$

defines the absolute choreographies. Similar to the three-vortex problem, one can show that there is a countable set of absolute choreographies, described by Goryachev's solution with different m and k.

Stability.   As a conclusion, let us discuss the stability of the specified periodic solutions.

Since the three-vortex problem is integrable, all the solutions of the reduced system (9) are periodic and stable. Yet it is easy to show that any (absolute or relative) choreography in this problem is neutrally stable under perturbations of the vortices' positions in the absolute space.

In the four-vortex problem, due to its non-integrability, the relative choreographies can be (exponentially) unstable. Yet, if a periodic solution of the reduced system (10) is stable, the corresponding choreographies are also neutrally stable in the absolute space. The numerical analysis of the multipliers of the periodic solutions that describe choreographies presented in Fig. **??** b, c shows that they are (exponentially) unstable.

References

[1] *H. Aref, N. Pomphrey.* Integrable and chaotic motions of four vortices. I – The case of identical vortices. Proc. Roy. Soc. London Ser. A. 1982. V. 380. P. 359–387.

[2] *A. V. Bolsinov, A. V. Borisov, I. S. Mamaev.* Lie algebras in vortex dynamics and celestial mechanics — IV. Reg. & Chaot. Dyn. 1999. V. 4. №1. P. 23–50.

[3] *A. V. Borisov, I. S. Mamaev.* Poisson structures and Lie algebras in Hamiltonian mechanics. Izhevsk: Udmurt University Press. 1999. (In Russian)

[4] *A. V. Borisov, I. S. Mamaev.* Mathematical methods in the dynamics of vortex structures. In: Fundamental and applied problems of the vortex theory, eds. A. V. Borisov, I. S. Mamaev, M. A. Sokolovsky. Moscow–Izhevsk: ICS. 2003. P. 17–178. (In Russian)

[5] *C. V. L. Charlier.* Die Mechanik des Himmels. Berlin: W. de Gruyter. 1927.

[6] *A. Chenciner, R. Montgomery.* A remarkable periodic solution of the three-body problem in the case of equal masses. Annals of Mathematics. 2000. V. 152. №3. P. 881–901.

[7] *A. Chenciner, J. Gerver, R. Montgomery, C. Simo.* Simple choreographies of N bodies: a preliminary study. In: Geometry, Mechanics, and Dynamics. Volume in honor of the 60th birthday of J. E Marsden, eds. P. Newton, P. Holmes, A. Weinstein. Springer. 2001.

[8] *D. N. Goryachev.* On certain cases of motion of rectilinear parallel vortices. Moscow University Printing House. 1898. (In Russian)

[9] *K. M. Khanin.* Quasi-periodic motions of vortex systems. Physica D. 1982. V. 4. P. 261–269.

[10] *V. V. Kozlov.* Symmetries, topology, and resonances in Hamiltonian mechanics. Izhevsk, Udmurt University Press. 1995. (In Russian)

[11] *C. C. Lim.* Graph theory and special class of symplectic transformations: the generalized Jacobi variables. J. Math. Phys. 1991. V. 32. №1. P. 1–7.

[12] *C. C. Lim.* A combinatorial perturbation method and whiskered tori in vortex dynamics. Physica D. 1993. V. 64. №163–184. P. 1–7.

[13] *J. Marsden, A. Weinstein.* Reduction of symplectic manifolds with symmetry. Rep. on Math. Phys. 1974. V. 5. №1. P. 121–130.

[14] *V. V. Meleshko, M. Yu. Konstantinov.* Dynamics of vortex structures. Kiev: Naukova Dumka. 1993. (In Russian)

[15] *J. K. Moser.* Lectures on Hamiltonian Systems. Mem. Amer. Math. Soc. 1968. V. 81. P. 1–60.

[16] *P. K. Newton.* The N-Vortex problem. Analytical Techniques. Springer. 2001.

[17] *C. Simo.* New families of solutions in N-body problems. Proc. ECM 2000, Barcelona. Progress in Mathematics. Birkhäuser Verlag. 2000. V. 201. P. 101–115.

[18] *J. L. Synge.* On the motion of three vortices. Can. J. Math. 1949. V. 1. P. 257–270.

[19] *J. J. Thomson.* The Corpuscular Theory of Matter. London and Tonbridge. 1907.

[20] *S. L. Ziglin.* Non-integrability of the problem of motion of four point vortices. Doklady AN SSSR. 1979. V. 250. №6. P. 1296–1300. (In Russian)